A greyness reduction framework for prediction of grey heterogeneous data

Chong Li[a,b*], Yingjie Yang[b], Sifeng Liu[c]

a. School of Economics & Management, Fuzhou University, Fuzhou, 350108, China
b. Centre for Computational Intelligence, De Montfort University, Leicester, LE1 9BH, UK
c. College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, China

ABSTRACT:

Existing operational rules of interval grey numbers do not make full use of possible background information when determining the interval boundaries, and this may result in inconsistent results if applying different logical operations. This paper finds that multiplication and division rules of interval grey numbers do not meet the calculation rule of inverse operators. Direct solution and inverse solution of the same interval grey number object may differ not only in numerical ranges but also in greyness degrees. To improve the accuracy of grey number calculation, new operational rules for multiplication and division of interval grey numbers are proposed. Then the traditional prediction modeling method of grey heterogeneous data is refined and expanded by integrating a greyness reduction preprocessing, which is based on the proposed calculation rules. Application of the expanded heterogeneous interval grey number prediction model to a stock replenishment scheduling problem in emergency rescue scenarios is included to illustrate the new operational rules of grey numbers and their application in prediction algorithm, and the proposed approach is compared with other existing methods to demonstrate its effectiveness.

Keywords: Operational rule; Greyness reduction; Grey interval number; Prediction model; Stock replenishment scheduling.

[*] Corresponding author. Tel: +86 (0)591 22866412
E-mail address: lichong@nuaa.edu.cn (C. Li), yyang@dmu.ac.uk (Y.J. Yang), sfliu@nuaa.edu.cn (S.F. Liu)
Address: School of Economics & Management, Fuzhou University, 2 Xue Yuan Rd., University Town, Fuzhou 350108, China.

1. Introduction

Grey prediction is an important part of the grey system theory. As an emerging decision-making tool for system evolution estimations or predictions, grey prediction theory has caught the attention of scholars and practitioners from various disciplines and been utilized in a wide variety of fields (Wu et al. 2015; Liu and Forrest, 2010). The main objectives of current researches related to grey prediction are: (1) The extension and modification of the classic GM(1,1) model and their practical applications (Tsaur, 2008; Tsaur, 2009; Xie and Liu, 2005). (2) The improvement of the background information and raw data used in developing a prediction model (Evans, 2014; Li, 2015; Wu et al. 2016). (3) Comparative assessment of grey model-based methods and common time series forecasting methods in different application fields (Bezuglov and Comert, 2016). (4) The combinations of grey models and other theoretical models to improve the accuracy of predictions (Guo et al. 2015; Ogiela et al. 2014; Wu et al. 2014).

The majority of these researches have focused on models based on analyzing the sequence of real numbers. It is more precisely in modern economic and social context that problems of economic development must be considered in scenarios with incomplete and uncertain information, such as disaster management and emergency rescue in some natural hazards and social emergencies. Models for real number sequence, usually requiring complete or accurate information, are no longer applicable (Liu and Forrest, 2010). When the historical sample of the object being forecasted is incomplete or the collected data is inaccurate, any pursuit of precise prediction models becomes meaningless. Thus, how to handle the various uncertainties encountered in forecasting becomes a major issue.

The characteristics of uncertainties in prediction problems can be classified into two groups: the incompleteness in information and the inaccuracies in data. The differences among the current uncertain theories usually rely on the characteristics of different uncertainties. Fuzzy theory and grey system theory have some commons in their basic number sets. On the one hand, several study show that, under certain conditions, a conventional triangular fuzzy number can be transformed into the same expression form as an interval grey number (Oztaysi, 2014; Tseng, 2009). This indicates that grey numbers can be considered as a special case of fuzzy number (Karmakar and Mujumdar, 2006; Shih et al. 2011). On the other hand, Yang and John (2012) pointed out that if characteristic function values of grey numbers are restricted within [0, 1], grey sets can be considered as an extension to fuzzy sets. Deschrijver and Kerre (2003) agreed with this opinion. Nevertheless, there are clear differences between fuzzy numbers and grey numbers.

In fuzzy mathematics, membership function is designed to describe the degree of an object belongs to a fuzzy number set. Membership functions are subjective and need to be defined before applying. In grey system theory, the possibility function describes the possibility that a grey number can take a certain value. A grey interval number describes a number with unknown position within a clear boundary. It is clear that the same interval number means quite different things to users of these two theories.

Concepts of grey number and greyness degree have been shown to be effective tools in describing systems in the content of uncertain information (Lin et al. 2008; Yang and John, 2012). It therefore may be reasonable to do more research on grey prediction models for grey sequence, especially for the grey interval number sequence. The development of grey number theory provides many useful calculation and analysis methods. The early algorithm of standard interval grey numbers, provided by Fang and Liu (2005), is still widely used today. Then Li et al. (2012) provided new grey number rules based on numerical coverage and their study extended the grey algebra system. Luo et al. (2018) redefined the greyness concept of some extended grey number and grey heterogeneous data sets. Then they designed a grey stochastic multiple criteria decision-making method based on the uncertain characteristics of grey heterogeneous data. Their work improved the performance of stochastic multiple criteria decisions. Recently, Ye et al. (2019) provided a new multi-angle information transformation method for interval grey numbers. This method can keep comprehensive information and optimize the resolution of traditional grey degree. Other new development in grey number calculations can be found in literatures (Guo et al. 2019; Xiao et al. 2020; Xiong et al. 2020).

Existing grey prediction models for grey interval numbers mainly focus on the *Homogeneous Data*. These models are based on the whitenizations of grey interval number sequence and on the greyness degrees of its elements (Salmeron and Papageorgiou, 2012; Yamaguchi et al. 2007). Some are based on the geometric features of interval grey numbers (Luo, 2009; Zeng et al. 2013), and the fluctuation range of the upper and lower boundaries (Li et al. 2007; Wu and Liu, 2009). Combination of models from other theory is also studied, such as the evolutionary analysis idea in grey hazy set (Li and Liu, 2009) and the classic statistical methods in normal distribution theory (Teng and Huang, 2013). Prediction model for *Heterogeneous Data* is first proposed by Zeng (2013). Both sequences of real numbers and interval grey numbers are considered in their study. They first build the discrete grey model based on the kernel sequence of heterogeneous data, to predict future kernels. Then applying the axiom of greyness degree not-reducing, they choose the biggest greyness value of the known sequence as the measure of the forecasted sequences. Though Zeng et al. (2015) later extended their heterogeneous data sequence model to including discrete grey numbers, their modeling logic did not change. The axiom of greyness degree not-reducing is applied in most of these studies. Although this axiom can be used fairly flexibly, it does not consider the evolution characteristics of data and thus cannot take full advantage of the background information. Sometimes it even leads to the greyness inflation of predicted data. Xiao et al.(2020) provided an effective method to transform the original multi-source heterogeneous data, consisting of real numbers, interval numbers and linguistic fuzzy numbers, into linguistic 2-tuples. This method improves the effectiveness of data integration.

This paper focuses on the available information implicit in the relationship and arithmetic logic between heterogeneous data. These implied background information are used to reduce the greyness degree of original known data series, so as to improve the quality of grey interval numbers in modeling. This improvement is embodied in the proposed new operational rules for multiplication and division of interval grey numbers. The greyness reduction preprocessing can also improve effectiveness of grey prediction model, reduce the range of the final predicted value and improve the comparability of decision alternatives that are based on prediction.

This paper is organized as follows. Section 2 presents related basic concepts and theories of grey systems. Section 3 describes the shortcomings of the commonly used multiplication and division rules of grey interval numbers, and proposes new operational rules to improve the accuracy of grey number calculation. Section 4 describes the discrete grey model for grey heterogeneous interval number, in which the greyness reduction preprocessing based on new operational rules is incorporated. In Section 5, we present the experimental results of the proposed heterogeneous interval grey number prediction model for inventory optimization in emergency rescue scenarios. Finally, Section 6 concludes this paper and describes future work.

2. Some basic concepts of grey systems

2.1. Basic concepts

Research objects in grey system theory are usually defined by grey numbers, grey equations, grey matrices, and so on. Among these mathematical forms, grey numbers are treated as basic "unit" or "cell" of grey systems. A grey number stands for an indeterminate number that takes its possible value within an interval or a general set of numbers (Liu and Forrest, 2010). Usually, the position of the exact value within the boundaries is unknown and cannot be described by the distribution functions in probability theory (Yang and John, 2003). A grey number is generally expressed using the symbol

"$\otimes$". When considering the relationship between its lower boundary and upper boundary, a grey number can be classified as: grey numbers with only a lower bound, grey numbers with only an upper bound, interval grey numbers, continuous and discrete grey numbers, black and white numbers, and so on. Among these categories, grey interval numbers are the most commonly studied.

Definition 1 - grey interval number (Liu et al. 2013). A grey interval number has both a lower a_k and an upper bound b_k, written as $\otimes_k \in [a_k, b_k]$, $a_k \le b_k$. The range of possible deviation of a grey interval number $\otimes_k$ is called the field of $\otimes_k$, usually measured by $d(\otimes_k) = b_k - a_k$.

Definition 2 - grey whitenization weight function (Liu and Xie, 2011). Function $f(x)$ is used to describe the preference which a grey number has over a potential value x it might take.

Definition 3 - kernel of a grey number (Liu and Fang, 2006). Based on background information and experience of researchers, value that is most representative of whitenization values of a grey interval number is called its kernel $\breve{\otimes}_k$. This value is usually a real number from whitenization function. In reality, the expected value of a grey interval number in its numeric field is usually defined as its kernel.

Definition 4 - the degree of greyness of a grey number (Liu and Lin, 2006). This concept expresses the degree of uncertainty of information in people's awareness of related number. Scholars have proposed various calculation methods for estimating this value. Professor Sifeng Liu introduced a definition of the degree of greyness, and it is the division of measures of the discourse field $\mu(\Omega)$ and the numeric field $\mu(\otimes)$:

$$g^0(\otimes) = \mu(\otimes) / \mu(\Omega) \tag{1}$$

and this definition satisfies the four axioms of degree of greyness (Liu et al. 2004).

When the studied number belongs to grey interval number: $\otimes_k \in [a_k, b_k]$, measure of $\otimes_k$'s field is usually represented by the length of its interval: $\mu(\otimes_k) = b_k - a_k$.

Axiom 1 (The axiom of non-decreasing greyness): rules of operations and algebraic system on grey numbers should follow the axiom of greyness degree not-decreasing. It means the degree of greyness of the calculated data should not be less than the larger one among the independent variables (Liu and Lin, 2006). In prediction models, especially models for a sequence of grey interval numbers, it is common to assign the greatest greyness value of the known sequence to the predicted numbers (e.g. Xie and Liu, 2005; Zeng et al., 2015).

Corollary 1 (Liu and Lin, 2006): The scalar multiplication of a grey number does not change its degree of greyness. This means, when a grey number $\otimes$ is multiplied by a real number k, the greyness of the original data satisfies: $g^0(\otimes) = g^0(k \cdot \otimes)$.

2.2. Calculations on kernels of grey interval numbers

Because many grey number algorithms are based on the kernel of grey numbers, the selection of kernel values will determine the final effectiveness of calculation. Liu and Forrest (2010) designed the following formulas for the kernel of grey interval numbers:

∗ When the distribution information of grey interval number $\otimes \in [a,b]$ is unknown,

(1) if $\otimes$ is continuous, then its kernel is: $\breve{\otimes} = (a+b)/2$

(2) if $\otimes$ is discrete, $\otimes_k \in [a_k, b_k]$, $a_k \le b_k$, then its kernel is $\breve{\otimes} = \frac{1}{n}\sum_{i=1}^{n} k_i$, where k_i are kernels of possible discrete numbers.

∗ When the distribution information of grey interval number $\otimes \in [a,b]$ is known,

(3) expectation value of this grey number is assigned to its kernel: $\breve{\otimes} = E(\otimes)$.

Latter, some studies suggested some new kernel formulas, which consider the geometric shape of numeric field. The following gives two examples:

(4) if the triangular whitenization weight function $f(\otimes)$ of a grey interval number is asymmetrical, then its kernel is the abscissa coordinates of the barycentric of the triangular whitenization weight function. For example, vertexes of $f(\otimes)$ are $A(a_k, 0)$, $B(b_k, 0)$ and $C(c_k, 1)$, G is the gravity center of triangle ΔABC , then the kernel of grey interval number $\otimes$ is: $\breve{\otimes}_k = X_G = (a_k + b_k + c_k)/3$

(5) If $\otimes$ has the trapezoid whitenization weight function, with vertexes $A(a_k, 0)$, $B(b_k, 0)$, $C(c_k, 1)$ and $D(d_k, 1)$, then its kernel is the abscissa coordinate of the gravity center of trapezoid $ABCD$, given by:

$$\breve{\otimes}_k = X_G = \frac{(2b_k - d_k + a_k + c_k)(d_k + c_k - a_k - b_k)/3 - (d_k - b_k)(a_k + b_k)}{(c_k - d_k)(b_k - a_k)}.$$

2.3. Basic operations of grey interval numbers

The classic operations of interval grey numbers are based on the minimum and maximum values obtained from operations on their boundaries. Assume that $\otimes_1 \in [a, b], a < b$ and $\otimes_2 \in [c, d], c < d$, then the operational rules are as follows (Liu et al. 2004):

(addition rule) $\otimes_1 + \otimes_2 \in [a + c,\ b + d]$

(subtraction rule) $\otimes_1 - \otimes_2 \in [a - d,\ b - c]$

(multiplication rule) $\otimes_1 \cdot \otimes_2 \in [\min\{ac, ad, bc, bd\},\ \max\{ac, ad, bc, bd\}]$

(division rule) $\frac{\otimes_1}{\otimes_2} \in \left[\min\left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\},\ \max\left\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\right\}\right]$ ($c \neq 0, d \neq 0, cd > 0$)

(scalar multiplication rule) $k \cdot \otimes_1 \in \begin{cases} [ka, kb], & if \quad k \geq 0 \\ [kb, ka], & if \quad k < 0 \end{cases}$

Given the kernel and degree of greyness, let symbol $*$ represent an operation between grey numbers, above operations can be redefined as:

$$\otimes_1 * \otimes_2 = \left(\breve{\otimes}_1 * \breve{\otimes}_2\right)_{g^0(\otimes_1) \vee g^0(\otimes_2)} \tag{2}$$

Yang and Liu (2011) extended the computational logic of operations in above formula, and provided simplified operations based on the concepts of kernel and degree of greyness. In their study, values of lower and upper limits of grey numbers are taken into account when deciding the degree of greyness of calculation result (please see (Yang and Liu, 2011) for more detailed description).

3. Greyness reduction operations of grey interval numbers

3.1. Comparison of direct and inverse solutions of grey interval numbers

Applying the classic rule of scalar multiplication in grey theory will sometimes produce confusing results. The following theorem gives an example of one possible situation.

Theorem 1. Let grey interval numbers share the same discourse field (or in other words, the same measure $\mu(\Omega)$), Corollary 1 is no longer valid (applicable) in some situations, if the measure of information field of grey interval number is defined by its length between upper and lower limits.

Proof. Let $\otimes_1$ and $\otimes_2$ be two grey interval numbers, $k(\neq 1)$ be a real number. Assume that the quantitative relationship between them satisfies: $\otimes_2 = k \cdot \otimes_1$. If applying the scalar multiplication rule described in section 2.3, the information field measures of two numbers are: $\mu(\otimes_2) = l(k \cdot b - k \cdot a) = k \cdot l(b-a) = k \cdot \mu(\otimes_1)$. The greyness degrees are: $g^0\left(\otimes_2\right) = k \cdot g^0\left(\otimes_1\right)$. If $k \neq 1$, we have $g^0\left(\otimes_2\right) \neq g^0\left(\otimes_1\right)$, but according to Corollary 1 it should be $g^0\left(\otimes_2\right) = g^0\left(\otimes_1\right)$. Results are conflicting and this finishes the proof of Theorem 1.

For the derivation of our greyness reduction operations, we first define two terms:

Definition 5 - inverse solution of grey number. An inverse solution of an unknown grey number (denoted as $\otimes_x$) is obtained by solving an explicit function of $\otimes_x$. More specifically, the left side of an explicit function contains only the number to be solved $\otimes_x$, while the right side is the explicit form of a function $f(\otimes_x)$. In the rest of this paper, the inverse solution of grey number $\otimes_x$ is denoted by $\otimes_x^1$.

Definition 6 - direct solution of grey number. A direct solution is obtained by directly solving some algebraic functions of the unknown grey number ($\otimes_x$), no matter whether $\otimes_x$ is explicitly or implicitly expressed in these functions. It is required that these algebraic functions follow the classic rules of operations of grey interval numbers (summarized in section 2.3). In the rest of this paper, the direct solution of grey number $\otimes_x$ is denoted by $\otimes_x^2$.

Example 1. $\otimes_1 \in (a,b)$ and $\otimes_2 \in (c,d)$ are two grey interval numbers whose information field have been identified. $\otimes_x$ is a grey interval number to be solved. Given that these numbers satisfy the numerical relationship: $\otimes_1 \cdot \otimes_x = \otimes_2$. Following the Definition 5, we have the inverse solution:

$$\otimes_x^1 = \otimes_2 / \otimes_1 = [\min(\frac{c}{a}, \frac{d}{a}, \frac{c}{b}, \frac{d}{b}), \max(\frac{c}{a}, \frac{d}{a}, \frac{c}{b}, \frac{d}{b})] \tag{3}$$

Following the Definition 6, we have the direct solution: $\otimes_x^2 \in (m,n)$, where m and n are determined by equations:

$$\begin{cases} \min(am, an, bm, bn) = c \\ \max(am, an, bm, bn) = d \end{cases} \tag{4}$$

Proposition 1. If numerical relationship between grey interval numbers can be expressed by addition or subtraction operations, there is no difference between inverse solution and direct solution, and these solutions share the same numerical ranges and degrees of greyness.

Proposition 2. If numerical relationship between grey interval numbers can be expressed by multiply or division operations, different calculation logic will result in different solutions and thus may in some circumstances affect their degrees of greyness, even when the measure of discourse field remains unchanged.

Example 2. $\otimes_1 \in (2,3), \otimes_2 \in (6,12)$ and $\otimes_x$ are three grey interval numbers satisfy $\otimes_1 \cdot \otimes_x = \otimes_2$. According to equation (3), the inverse solution of $\otimes_x$ is $\otimes_x^1 \in (2,6)$; according to equation (4), the direct solution of $\otimes_x$ is $\otimes_x^2 \in (3,4)$. Obviously, without changing the measure of the discourse field, value ranges and greyness degree values of two solutions are quite different.

In the rest of this paper, we assume that all grey interval numbers share the same discourse field and information field be defined by the length between upper and lower limits of grey numbers. Let $\otimes_k$ and $\otimes_{1-k}, (k = 0\ or\ 1)$ be two grey interval numbers whose information fields have been identified. $\otimes_x$ is a grey interval number to be solved. For subsequent comparative analysis, we summarize all the possible inverse solutions and direct solutions with respect of multiply and division operations of these grey interval numbers in following statements.

Mathematical form 1 - When the numerical relationship between three grey interval numbers is established by division operation, expressed mathematically as $\otimes_x / \otimes_1 = \otimes_0$. Consider following 6 cases:

Case 1. $\otimes_k \in [a,b], 0 < a < b,\ \otimes_{1-k} \in [c,d], 0 < c < d$. We have:

When $k=1$, $\otimes_x^1 = [ac, bd]$, $\otimes_x^2 = [bc, ad]$; when $k=0$, $\otimes_x^1 = [ac, bd]$, $\otimes_x^2 = [ad, bc]$. According to assumptions and definition 4, it shows that under the same k, degrees of greyness satisfy $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 2. $\otimes_k \in [a,b], a < b < 0,\ \otimes_{1-k} \in [c,d], c < d < 0$. We have:

When $k=1$, $\otimes_x^1 = [bd, ac], \otimes_x^2 = [ad, bc]$; when $k=0$, $\otimes_x^1 = [bd, ac]$, $\otimes_x^2 = [bc, ad]$. In this case, it shows that under the same k value, the degrees of greyness satisfy $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 3. $\otimes_k \in [a,b], 0 < a < b,\ \otimes_{1-k} \in [c,d], c < d < 0$. We have:

When $k=1$, $\otimes_x^1 = [bc, ad], \otimes_x^2 = [ac, bd]$; when $k=0$, $\otimes_x^1 = [bc, ad], \otimes_x^2 = [bd, ac]$. In this case, it shows that under the same k value the degrees of greyness $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 4. $\otimes_k \in [a,b], 0 < a < b,\ \otimes_{1-k} \in [c,d], c < 0 < d$. We have:

When $k=1$, $\otimes_x^1 = [bc, bd], \otimes_x^2 = [ac, ad]$; when $k=0$, $\otimes_x^1 = [bc, bd]$, it has no solution to $\otimes_x^2$ in this case. When there is no solution, we can regard it as a definitive value, expressed as $\otimes_x^2 = [null]$. It is clear that $g^\circ([null]) = 0$. So in this case, under the same k value we have the degrees of greyness $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 5. $\otimes_k \in [a,b], a < b < 0,\ \otimes_{1-k} \in [c,d], c < 0 < d$. We have:

When $k=1, \otimes_x^1 = [ad, ac], \otimes_x^2 = [bd, bc]$; when $k=0$, $\otimes_x^1 = [ad, ac], \otimes_x^2 = [null]$. Similar to the explanation in Case 4, under the same k value, we have $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 6. $\otimes_k \in [a,b], a < 0 < b,\ \otimes_{1-k} \in [c,d], c < 0 < d$. We have:

Unlike the above five cases, both inverse solution and direct solution of parameter $\otimes_x$ can not be accurately determined without additional information on its value range. Even given the information field of $\otimes_x^2$, its upper and lower boundary values may not be unique and multiple solutions with different positive and negative phrases may exist. So in this paper we do not discuss the details of different logical operations in this situation.

Summarize case 1 to case 5 reported above, we have the following theorem:

Theorem 2. Let $\otimes_x^1$ and $\otimes_x^2$ be the reverse and direct solution of grey interval number $\otimes_x$ expressed in the algebraic function with division operation: $\otimes_x / \otimes_1 = \otimes_0$. In all five cases (case 1 to 5), solution $\otimes_x^1$ and $\otimes_x^2$ have different information field, degrees of greyness of these solutions all satisfy: $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Mathematical form 2 - When the numerical relationship between three grey interval numbers is established by multiply operation, expressed mathematically as $\otimes_x \cdot \otimes_1 = \otimes_0$. Because the multiplication rule of the classic grey interval numbers satisfies the commutative principle for multiplication, analysis result can be applied to $\otimes_1 \cdot \otimes_x = \otimes_0$. Similar to analysis in Mathematical form 1, we consider following 6 cases:

Case 7. $\otimes_k \in [a,b], 0 < a < b,\ \otimes_{1-k} \in [c,d], 0 < c < d$. We have:

When $k=1$, $\otimes_x^1 = [c/b, d/a]$, $\otimes_x^2 = [c/a, d/b]$; when $k=0$, $\otimes_x^1 = [a/d, b/c]$, $\otimes_x^2 = [a/c, b/d]$. In this case, it shows that under the same k value the degrees of greyness satisfy: $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 8. $\otimes_k \in [a,b], a<b<0,\ \otimes_{1-k} \in [c,d], c<d<0$. We have:

When $k=1,\ \otimes_x^1 = [d/a, c/b],\ \otimes_x^2 = [d/b, c/a]$; when $k=0,\ \otimes_x^1 = [b/c, a/d]$, $\otimes_x^2 = [b/d, a/c]$. Under the same k value, the greyness degrees of different solutions satisfy: $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 9. $\otimes_k \in [a,b], 0<a<b,\ \otimes_{1-k} \in [c,d], c<d<0$. We have:

When $k=1,\ \otimes_x^1 = [c/a, d/b],\ \otimes_x^2 = [c/b, d/a]$; when $k=0,\ \otimes_x^1 = [b/d, a/c]$, $\otimes_x^2 = [b/c, a/d]$. In this case, it shows that under the same k value the degrees of greyness $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 10. $\otimes_k \in [a,b], 0<a<b,\ \otimes_{1-k} \in [c,d], c<0<d$. We have:

When $k=1,\ \otimes_x^1 = [c/a, d/a],\ \otimes_x^2 = [c/b, d/b]$; when $k=0,\ \otimes_x^1 = [b/c, b/d]$, and there is no solution to $\otimes_x^2$ in this scenario. Similar to analysis in Case 4 with division form, we have $\otimes_x^2 = [null]$ and the inequality of greyness degrees is: $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 11. $\otimes_k \in [a,b], a<b<0,\ \otimes_{1-k} \in [c,d], c<0<d$. We have:

When $k=1$, $\otimes_x^1 = [d/b, c/b], \otimes_x^2 = [d/a, c/a]$; when $k=0$, $\otimes_x^1 = [a/d, a/c]$, $\otimes_x^2 = [null]$. Under the same k value, we have $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

Case 12. $\otimes_k \in [a,b], a<0<b,\ \otimes_{1-k} \in [c,d], c<0<d$.

Similar to the scenario described in Case 6 with a division operation, values of both inverse solution $\otimes_x^1$ and direct solution $\otimes_x^2$ of parameter $\otimes_x$ are depended on the information field, or the upper and lower boundaries. We follow the same approach as in Case 6, namely, we do not discuss in detail the possible solutions in this scenario.

The following theorem summarizes the performance of case 7 to case 11.

Theorem 3. Let $\otimes_x^1$ and $\otimes_x^2$ be the reverse and direct solution of grey interval number $\otimes_x$ expressed in the algebraic function with multiplication operation: $\otimes_x \cdot \otimes_1 = \otimes_0$. Solutions $\otimes_x^1$ and $\otimes_x^2$ have different information field, and their degrees of solution greyness in case 7 to case 11 all satisfy: $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$. This conclusion also applies to solutions in $\otimes_1 \cdot \otimes_x = \otimes_0$.

Theorem 2 and 3 provide some useful indications for calculations of grey interval numbers. We summarize them in the following theorem:

Theorem 4. The rules of multiplication and division of interval grey numbers do not meet the calculation rule of inverse operators. The reciprocal relationship between multiplication operation and division operation is not applicable to the algebraic function containing unknown grey numbers. Let $\otimes_k$ and $\otimes_{1-k}$ $(k=0\ or\ 1)$ be two grey numbers whose information fields have been identified, and $\otimes_x$ be a grey interval number to be solved. Assume that all grey interval numbers share the same measure of discourse field, and the measure of information field be defined by the length between upper and lower limits of grey numbers. When the information field of $\otimes_k$ and $\otimes_{1-k}$ $(k=0\ or\ 1)$ do not synchronized vary across positive and negative values, namely situations described in Case 6 and 12 are excluded from this argument, greyness degree of direct solution based on the original numerical function is less than that based on the explicit reformulation of the original function: $g^\circ(\otimes_x^1) > g^\circ(\otimes_x^2)$.

3.2. Greyness reduction rules for multiply and division operations

In this section we discuss new operational rules for multiplication and division of interval grey numbers. Based on the discussion in Section 3.1, the following rules give a simplified method to improve the calculation accuracy of grey interval numbers:

Assume that $\otimes_k \in [a,b], a<b,\ \otimes_{1-k} \in [c,d], c<d,\ (k=0\ or\ 1)$ are two grey interval numbers whose upper and lower boundaries have been identified and do not synchronized vary across positive and negative value. $\otimes_x$ is a grey interval number to be solved.

Rule 1 – *rule of division operation*: if measurements of additional background information give directly a numerical relationship between grey numbers in form of $\otimes_x / \otimes_1 = \otimes_0$, the solution of $\otimes_x$ is defined as follows:

$$\{\otimes_x \mid k=1\} = \begin{cases} [bc, ad], & when\ \ 0<a<b\ and\ 0<c<d \\ [ad, bc], & when\ \ a<b<0\ and\ c<d<0 \\ [ac, bd], & when\ \ 0<a<b\ and\ c<d<0 \\ [ac, ad], & when\ \ 0<a<b\ and\ c<0<d \\ [bd, bc], & when\ \ a<b<0\ and\ c<0<d \end{cases} \tag{5}$$

$$\{\otimes_x \mid k=0\} = \begin{cases} [ad, bc], & when\ \ 0<a<b\ and\ 0<c<d \\ [bc, ad], & when\ \ a<b<0\ and\ c<d<0 \\ [bd, ac], & when\ \ 0<a<b\ and\ c<d<0 \\ [null], & when\ \ 0<a<b\ and\ c<0<d \\ [null], & when\ \ a<b<0\ and\ c<0<d \end{cases} \tag{6}$$

Rule 2 – *rule of multiplication operation*: if measurements of additional background information give directly a numerical relationship between grey numbers in form of $\otimes_x \cdot \otimes_1 = \otimes_0\ (or\ \ \otimes_1 \cdot \otimes_x = \otimes_0)$, solution of $\otimes_x$ is defined as follows:

$$\{\otimes_x \mid k=1\} = \begin{cases} [c/a,\ d/b], & when\ \ 0<a<b\ and\ 0<c<d \\ [d/b,\ c/a], & when\ \ a<b<0\ and\ c<d<0 \\ [c/b,\ d/a], & when\ \ 0<a<b\ and\ c<d<0 \\ [c/b,\ d/b], & when\ \ 0<a<b\ and\ c<0<d \\ [d/a,\ c/a], & when\ \ a<b<0\ and\ c<0<d \end{cases} \tag{7}$$

$$\{\otimes_x \mid k=0\} = \begin{cases} [a/c,\ b/d], & when\ \ 0<a<b\ and\ 0<c<d \\ [b/d,\ a/c], & when\ \ a<b<0\ and\ c<d<0 \\ [b/c,\ a/d], & when\ \ 0<a<b\ and\ c<d<0 \\ [null], & when\ \ 0<a<b\ and\ c<0<d \\ [null], & when\ \ a<b<0\ and\ c<0<d \end{cases} \tag{8}$$

Equations (5)-(8) provide simple operational rules for the unknown grey interval numbers in multiplication and division operations. Comparing these rules with the findings in Section 3.1, it is obvious that the proposed rules of division and multiplication operations are based on the direct solutions for the corresponding instances. According to Theorem 4, results from Equations (5)-(8) enjoy lower degrees of greyness and thus can improve the effectiveness of subsequent calculation.

4. Greyness reduction framework for grey heterogeneous data prediction

To improve the accuracy of forecasting method for grey numbers, appropriate data preprocessing is necessary. As mentioned in Section 1, the traditional discrete grey number prediction models might lead to the greyness inflation of predicted data and undermine the effectiveness and practicality of prediction results. In order to remedy this defect, this paper proposes a framework for the prediction of grey heterogeneous data. Application of the greyness reduction rules of grey number operations is integrated as part of data preprocessing. The proposed prediction framework consists of two phases: (1) the data preprocessing based on the designed greyness reduction rules for multiply and division operations and (2) the discrete grey model based on the greyness reduced sequences.

The following steps discuss how this proposed framework be used in the prediction of grey interval number sequence, which bears a multiplicative relationship to other heterogeneous data series. Applying the framework to grey number sequences with a division relationship between them, will be presented in Section 5.

Assume that $\otimes_X = (\otimes_x(1), \otimes_x(2), \dots \otimes_x(n))$, $\otimes_V = (\otimes_v(1), \otimes_v(2), \dots \otimes_v(n))$ and $\otimes_Z = (\otimes_z(1), \otimes_z(2), \dots \otimes_z(n))$ are three grey sequences, in which the upper and lower limits of elements can be obtained by historical data, expressed as $\otimes_x(i) \in \left[x_i^-, x_i^+\right], \otimes_v(i) \in \left[v_i^-, v_i^+\right]$, $\otimes_z(i) \in \left[z_i^-, z_i^+\right], (i = 1 \dots n)$. We assume that additional information reveals a multiplicative relationship between elements of these sequences: $\otimes_x(i) \cdot \otimes_v(i) = \otimes_z(i)$. Let $\hat{\otimes}_x^*\left(n+k\right) \in [\hat{x}_{n+k}^+, \hat{x}_{n+k}^-], (k = 1, 2 \dots m)$ stand for the trend of the original sequence $\otimes_x$, where the predicted values $\hat{x}_{n+k}^+$ and $\hat{x}_{n+k}^-$, $(k = 1, 2 \dots m)$ are obtained through following steps:

Step 1: Regarding $\otimes_x(i), i = 1 \dots n$ as unknown grey interval numbers. Considering their mathematical relationship with other two independent parameters, expressed by $\otimes_x(i) \cdot \otimes_v(i) = \otimes_z(i)$, we directly apply an appropriate expression from Equation (7) or (8) to obtain the direct solution $\otimes_x^2(i)$.

Step 2: Provided the accuracy of the mathematical expression of numerical relationship between grey interval numbers is guaranteed and the quality of the raw data is good, there should be an intersection between the value range of raw data and the calculated result which is based on other heterogeneous data. So, comparing the information fields of $\otimes_x^2(i)$ with original value $\otimes_x(i), (i = 1 \dots n)$, the intersection between them is assigned to $\otimes_x^*\left(i\right)$. It is clear that $g^\circ\left(\otimes_x^*(i)\right) < g^\circ\left(\otimes_x(i)\right)$. Here, we get the greyness reduced sequence $\otimes_{\mathrm{x}}^* = \left(\otimes_x^*(1), \otimes_x^*(2), \dots \otimes_x^*(n)\right)$, $\otimes_x^*(i) \in \left[x_i^{*-}, x_i^{*+}\right]$, and can further obtain its corresponding kernel sequence $\breve{\otimes}_{\mathrm{x}}^* = \left(\breve{\otimes}_x^*(1), \breve{\otimes}_x^*(2), \dots, \breve{\otimes}_x^*(n)\right)$ and the maximum value of greyness degree of its elements, denoted as $G^\circ(\otimes_X^*) = g^\circ(\otimes_x^*(h)) = \max\left\{g^\circ(\otimes_x^*(i)) \middle| i = 1, 2 \dots n\right\}$. According to the axiom of greyness degree not-reducing, this paper simplifies the design of greyness degree of predicted data $\hat{\otimes}_x^*\left(n+k\right)$ as $g^\circ(\hat{\otimes}_x^*(n+k)) = G^\circ(\otimes_X^*)$, $k = 1, \dots m$.

Step 3: To obtain the predicted kernel sequence $\hat{\breve{\otimes}}_X^*$, the discrete grey prediction model proposed by Zeng et al. (2015) is applied to the greyness reduced sequence $\otimes_{\mathrm{x}}^*$ obtained in Step 2. The value of the predicted kernel sequence $\hat{\breve{\otimes}}_X^* = \left(\hat{\breve{\otimes}}_x^*(1), \dots \hat{\breve{\otimes}}_x^*(n), \hat{\breve{\otimes}}_x^*(n+1), \dots \hat{\breve{\otimes}}_x^*(n+k)\right)$, $(k = 1, \dots m)$ is defined as follows:

$$\hat{\breve{\otimes}}_x^*\left(n+k\right) = [(\beta_1 - 1) \cdot \breve{\otimes}_x^{*(1)}(1) + \beta_2] \cdot \beta_1^{n+k\text{-}2}, \quad k = 1, \dots m \tag{9}$$

where $\breve{\otimes}_x^{*(1)}(i)$ is the 1-AGO data from sequence $\breve{\otimes}_X^*$, parameters β_1 and β_2 are based on a least squares estimation approach, satisfying:

$$\left(\beta_1, \beta_2\right)^T = \left(B^T B\right)^{-1} B^T Y,$$

$$B=\begin{bmatrix} \breve{\otimes}_x^{*(1)}(1) & 1 \\ \breve{\otimes}_x^{*(1)}(2) & 1 \\ \vdots & \vdots \\ \breve{\otimes}_x^{*(1)}(n-1) & 1 \end{bmatrix},\ Y=\begin{bmatrix} \breve{\otimes}_x^{*(1)}(2) \\ \breve{\otimes}_x^{*(1)}(3) \\ \vdots \\ \breve{\otimes}_x^{*(1)}(n) \end{bmatrix}$$

Step 4: Combine the predicted kernels from Step 3 and the maximum value of greyness degree from Step 2, the upper and lower boundaries of predicted grey interval numbers $\hat{\otimes}_x^*(n+k)\in[\hat{x}_{n+k}^+,\hat{x}_{n+k}^-]$ can therefore be determined:

$$\begin{cases} \hat{x}_{n+k}^+ =[(\beta_1-1)\cdot\breve{\otimes}_x^{*(1)}(1)+\beta_2]\cdot\beta_1^{n+k\text{-}2}+\dfrac{x_h^{*+}-x_h^{*-}}{2} \\ \hat{x}_{n+k}^- =[(\beta_1-1)\cdot\breve{\otimes}_x^{*(1)}(1)+\beta_2]\cdot\beta_1^{n+k\text{-}2}-\dfrac{x_h^{*+}-x_h^{*-}}{2} \end{cases} \tag{10}$$

It is obvious that in this framework the phase of greyness reduction preprocessing consists of Step 1 and 2, while the phase of prediction consists of the last two steps. It is important to note that the prediction method used in Step 3 can be replaced by some other grey prediction model that could provide a necessary alternative for special applications. This will reinforce the flexibility and applicability of the proposed framework.

5. Framework application

In order to highlight the effectiveness and practicability of the proposed prediction framework, we consider an inventory optimization problem in emergency rescue scenarios.

Emergency supply chain management, especially those for relief supplies, plays a key role in emergency rescue operations. An efficient supply strategy usually depends on timely and accurate availability of information from both the supply side and the demand side. However, in most emergency rescue scenarios, quantity and quality of information cannot be guaranteed. For example, in the emergency response after earthquake disasters, information about the possibility of aftershocks or other subsequent geological events, the number of casualties and the affected area, cannot be obtained directly and accurately. In addition, many types of disasters occur far less frequently, which makes the obtained historical data less reliable. Under these situations, the accurate information about the demand for emergency relief supplies is hard to obtain. How to use the limited and imprecise data to design a timely and efficient replenishment strategy is very important.

Though it is difficult to access comprehensive and accurate information, small sample data are still available through some multi-source information collection techniques, and these data usually include characteristics of grey information. In this case study, we focus on the replenishment intervals, which show unexpected fluctuations caused by the uncertain traffic conditions and production capacity of upstream suppliers. Based on the greyness reduction framework described in Section 4, we first reduce the greyness degree of the replenishment interval time sequence. Then an inventory optimization model is established based on the predicted greyness-reduced replenishment intervals. From this model, the optimal replenishment quantity of emergency food can be calculated, together with the expected maximum profits.

5.1. Model assumption and symbol description

In this case study, we assume:

(1) Replenishment intervals of emergency food are mainly determined by the production capacity of upstream suppliers and the traffic conditions between suppliers and stock location. In this case study, all these factors are expressed by grey interval numbers.

(2) Considering the uncertainty of demand in emergency rescue and the various factors that affect inventory levels, inventory shortage and backlogging are allowed but they impose significant extra costs on emergency inventory management.

(3) To simplify our analysis, we do not include the transportation costs in the proposed inventory optimization model. This simplification will not significantly affect the conclusions from model analysis, because the transportation costs are mainly driven by the transport distance when the fluctuation of replenishment batch is relatively small.

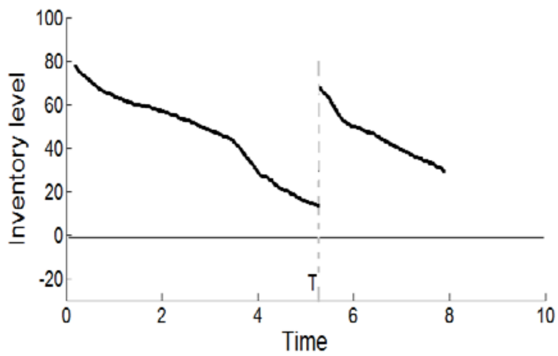
Model parameters are summarized in Table 1.

Table 1
Symbol description.

Symbol	Symbol Description
S	the pure stock at the beginning of each stock replenishment period.
T	replenishment interval.
r	demand rate of emergency food from downstream customers (unit: pieces/time).
a	food deterioration rate - proportion of deteriorating items in total inventory quantity per unit time (unit: %/time).
t_s	time point when out-of-stock occurs.
$I(t)$	inventory level at time point t .
c_1	unit purchase price of emergency food.
c_2	unit selling price of emergency food.
p_1	inventory cost per unit item and unit time.
p_2	cost per unit of goods out-of-stock.
x	service level, $x \in (0,1)$. It means that time proportion of out-of-stock state during replenishment intervals should below this value.
θ	traffic condition, θ is a grey interval number. The larger this value, the worse the traffic condition. It represents the normal traffic condition when $\theta = 1$.
h	production capability of upstream suppliers; similar to parameter θ , it is a grey interval number. The larger this value, the worse the production capability.
tt	the average length of time intervals between adjacent inventory replenishments, in conditions of normal supplying capacity $(h = 1)$ and transportation $(\theta = 1)$.

5.2. Model construction

According to values of replenishment time interval T and the possible time point of out-of-stock t_s , the dynamics of inventory can be divided into two cases: (1) $T \le t_s$, it means no shortage occurs (see Fig.1); (2) $T > t_s$, it means the replenishment takes place after the inventory is exhausted at time t_s (see Fig.2). Considering costs from inventory shortage, it is clear that two cases have different cost components.



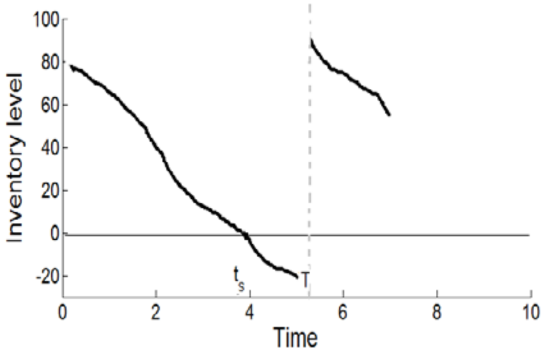


Fig.1. Inventory level without stock-out situation ($T \le t_s$).

Fig.2. Inventory level with stock-out situation ($T > t_s$).

According to the definition of model parameters in Table 1, dynamics of inventory level between two adjacent replenishments can be expressed as:

$$\begin{cases} \dfrac{dI(t)}{dt} + aI(t) + r = 0 & 0 \le t < t_s \\ \dfrac{dI(t)}{dt} + r = 0 & t_s \le t \le T \end{cases} \tag{11}$$

Solving the above differential equations, inventory level at time t is obtained:

$$I(t) = \begin{cases} I_1(t) = \dfrac{r}{a}\left(e^{a(t_s - t)} - 1\right) & 0 \le t < t_s \\ I_2(t) = r\left(t_s - t\right) & t_s \le t \le T \end{cases} \tag{12}$$

The total replenishment quantity of emergency food at the beginning of a new inventory cycle minus the cumulative unsatisfied demands at the end of the previous inventory cycle equals to the pure initial inventory level S, which can be derived from equation (12):

$$S = I_1(0) = \frac{r}{a} \cdot (e^{(at_s)} - 1) \tag{13}$$

Though parameter S is chosen as the decision variable in our model, above expression also reveals that it has a close relationship with parameter t_s. Their relationship can be rewritten as:

$$t_s = \frac{1}{a} \ln\left(1 + \frac{a}{r} S\right) \tag{14}$$

Case-1: When there is no stock-out, namely $T \le t_s$

The cumulative inventory of emergency food associated with inventory cost between two adjacent replenishments is:

$$\Phi_1 = \int_0^T \frac{r}{a}\left(e^{a(t_s - t)} - 1\right) dt \tag{15}$$

The corresponding inventory cost equals: $p_1 \cdot \Phi_1$.

The total sales of emergency food during the same period is:

$$Q_1 = rT \tag{16}$$

and the sales revenue equals: $c_2 \cdot Q_1$.

Case-2: When stock-out occurs, namely $T > t_s$

The cumulative inventory associated with inventory cost between two adjacent replenishments is:

$$\Phi_2 = \int_0^{t_s} \frac{r}{a}\left(e^{a(t_s - t)} - 1\right)dt \tag{17}$$

and its corresponding inventory cost equals: $p_1 \cdot \Phi_2$.

When $T > t_s$, it means the stock will be exhausted before new replenishment order arrives. So in this situation, the amount of sales of emergency food equals to the initial pure inventory level:

$$Q_2 = \frac{r}{a}\ln\cdot\left(\frac{a}{r}S + 1\right) \tag{18}$$

The sales revenue in this case equals: $c_2 \cdot Q_2$.

The cumulative unsatisfied demands between two adjacent replenishments is:

$$\Phi_3 = r \cdot \left(T - t_s\right) \tag{19}$$

The shortage cost is a function of the excess of demand over supply at the end of an inventory cycle. Therefore, it equals: $p_2 \cdot \Phi_3$.

In order to maximize profits over a replenishment period of emergency food supplies, it is necessary to develop a dynamic profit model with respect to the possible costs and benefits described above. The inventory optimization model for emergency food replenishment can be formulated as follows:

$$\begin{aligned} \max \quad & W\left(S\right) = \delta \cdot \left[c_2 \cdot Q_1 - p_1 \cdot \Phi_1\right] + \left(1-\delta\right) \cdot \left[c_2 \cdot Q_2 - p_1 \cdot \Phi_2 - p_2 \cdot \Phi_3\right] - c_1 \cdot S \\ s.t. \quad & \frac{\left(1-\delta\right) \cdot \Phi_3}{r \cdot T} \le x \end{aligned} \tag{20}$$

where δ is a symbolic parameter and it represents whether the stock shortage occurs or not:

$$\delta = \begin{cases} 1, & for \;\; T \le t_s \\ 0, & for \;\; T > t_s \end{cases}$$

Note that this symbolic parameter seems to depend on the comparison of shortage time and replenishment time, but in fact it is determined by the value of decision variable S. Quantitative relationship between these parameters is described by Equation (13). It means that inventory replenishment decision variable S will not only affect costs of procurement and inventory holdings, but also determine the indirect opportunity cost of losing revenue.

5.3. Model solution

Another important parameter in model (20) is the replenishment interval T. As that described at the beginning of Section 5, this data is assumed to be a grey interval number. In reality, a worse traffic condition usually leads to longer replenishment cycle, while productivity and delivery performance of upper suppliers also have impact on the length of inventory replenishment. We express their relationship by: $\theta = T/(tt \cdot h)$, where parameters T, θ and h are defined as sequences with grey interval numbers. The historical ranges of these values are set:

$$\otimes_{\mathrm{T}} = \left(\otimes_T(1),\ldots\otimes_T(5)\right) = \left([32, 34],[35, 38],[36.8, 39.0],[35.2, 37.0],[34.9, 37.5]\right),$$
$$\otimes_{\theta} = \left(\otimes_\theta(1),\ldots\otimes_\theta(5)\right) = \left([1.03, 1.07],[1.08, 1.13],[1.09, 1.16],[1.05, 1.13],[1.07, 1.12]\right),$$
$$\otimes_{\mathrm{h}} = \left(\otimes_h(1),\ldots\otimes_h(5)\right) = \left([1.08, 1.09],[1.10, 1.11],[1.15, 1.17],[1.09, 1.10],[1.08,1.09]\right).$$

Values of other model parameters used in simulation are shown in Table 2.

Table 2
Model parameter values.

Parameter	r	a	c_1	c_2	p_1	p_2	x	tt
Value	5	0.013	28	80	0.5	7	0.2	30

According to the numerical relationship between parameters T, θ and h, the greyness reduced sequence of replenishment intervals can be directly obtained based on the rule of division operation expressed in Equation (5). Here we get:

$$\otimes_{\mathrm{T}}^{*} = \left(\otimes_T^{*}(1),\ldots\otimes_T^{*}(5)\right)$$
$$= \left([33.681, 34.000],[35.964, 37.290],[38.259, 39.000],[35.200, 36.951],[34.989, 36.288]\right)$$

Based on sequence $\otimes_{\mathrm{T}}^{*}$, the evolution trend of replenishment intervals is then predicted and used to solve the remaining optimization problem. Simulation results based on both the proposed prediction framework of this paper and the widely used discrete grey number prediction model (Xie and Liu, 2005; Zeng et al., 2015) are depicted in Fig.3 and Fig.4, respectively.

Fig.3 shows the upper and lower boundaries of replenishment interval T, based on its original data as well as on the greyness-reduced values. Kernels of these values are shown by dash-dot lines. As can be observed from Fig.3, compared with its raw data, value range of greyness-reduced T based on parameters θ and h is narrower than on the direct sequence data. The same pattern can be found in the predicted sequences. Even though there is a small forecasted area based on the greyness-reduced samples exceeding the lower range of that based on the original data, its kernels and most kernel-based deviation ranges (the light blue area in Fig.3) still locate inside of the effective prediction area. The forecasted values based on greyness reduced replenishment intervals (denoted as T_1), have relatively smaller greyness degrees than those based on original data (denoted as T_0), namely $g^{\circ}\left(T_3\right) < g^{\circ}\left(T_2\right)$. This means that extra information provided by the numerical relationship between variables can be used to reduce the uncertainties in predictions.

Fig.4 shows the value ranges of pure profit W of emergency food supplies in different replenishment periods. The upper boundaries of profit W are obtained based on the optimized replenishment quantity from inventory optimization model (20), where parameter T is represented by its upper boundary value, namely T_i^{+}. The lower boundaries of W are obtained based on lower boundary values T_i^{-}. As can be observed from Fig.4, changing the greyness degree values of raw data have a significant effect on the value of inventory policy parameter S and thus affect the final profits W. It is clear that replenishment optimization based on the greyness-reduced data is less sensitive to the uncertainties of historical replenishment intervals. In practice, the predicted pure profit with lower uncertainties usually is preferred by decision makers, for it helps them to make clear strategic decisions about where should be their next focus in allocating resources.

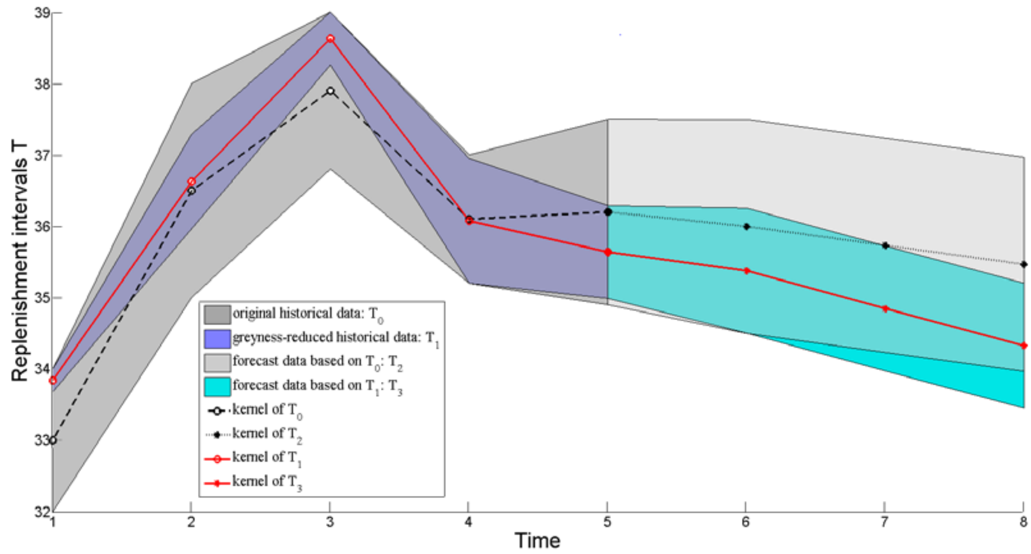


Fig.3. Replenishment intervals based on original and greyness reduced data.

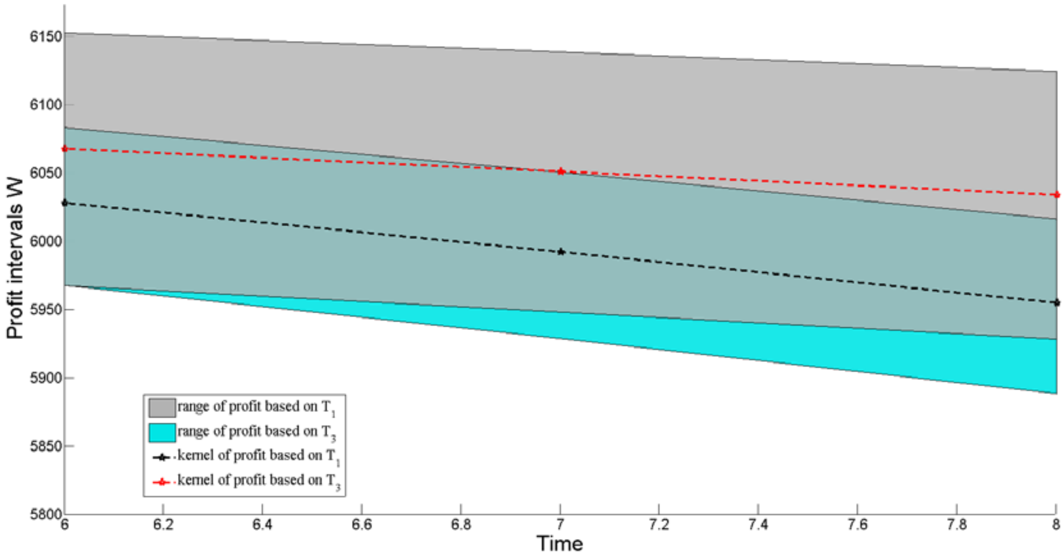


Fig.4. The expected ranges of pure profit based on different predicted replenishment intervals.

The above case study result shows the feasibility of the proposed greyness reduction method but it still needs future tests. One important test is the sensitivity analysis. If the proposed prediction framework based on the new greyness reduction rules also shows robust performance in a wide range of sample deviation conditions, the applicability of the proposed model will be extended. In the following paragraphs, we use the same inventory optimization problem to test the robustness of the proposed method.

Three different parameter setting scenarios are considered: first, the production capability of upstream suppliers parameter h presents a smoother fluctuations than the previous one; second, h presents a monotonous downward trend (i.e. upstream suppliers gradually restore normal supply); third, the traffic condition parameter θ shows a monotonous downward trend (i.e. the traffic obstructions are gradually cleared). These new parameter settings are summarized in Table 3 and other modeling parameters are kept the same as those in the previous study. Results of three sub-cases are shown in

Figs. 5 to 10, respectively.

Table 3
Parameter value adjustments used in three sub-cases.

Parameter		New series values
Sub-case 1	h	$\otimes_h = ([1.063, 1.068],[1.052, 1.060],[1.062, 1.063],[1.059, 1.065],[1.052, 1.063])$.
Sub-case 2	h	$\otimes_h = ([1.080, 1.090],[1.078, 1.085],[1.075, 1.080],[1.065, 1.072],[1.045, 1.060])$.
Sub-case 3	θ	$\otimes_\theta = ([1.037, 1.130],[1.037, 1.093],[1.035, 1.089],[1.026, 1.089],[1.003, 1.085])$.

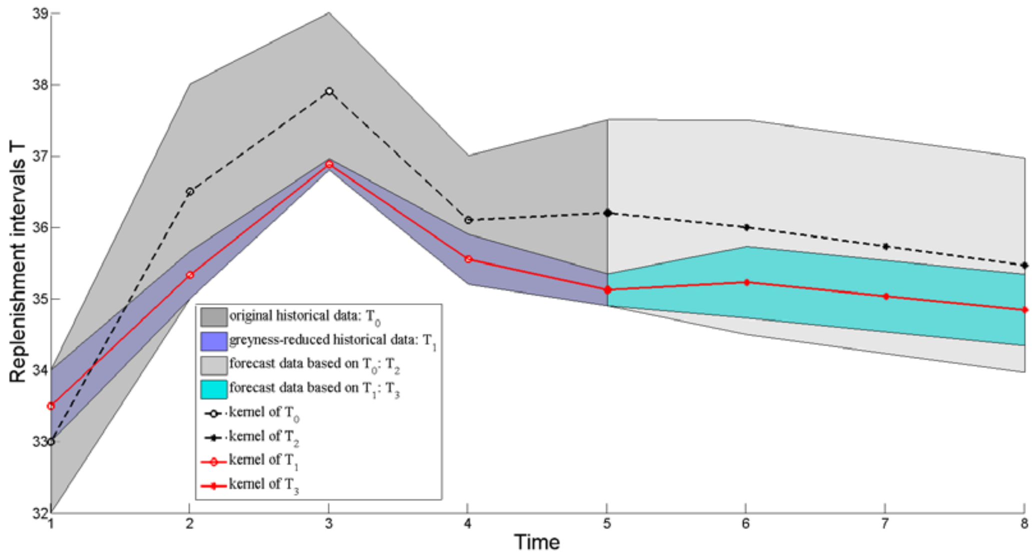


Fig.5. Replenishment intervals in Sub-case 1.

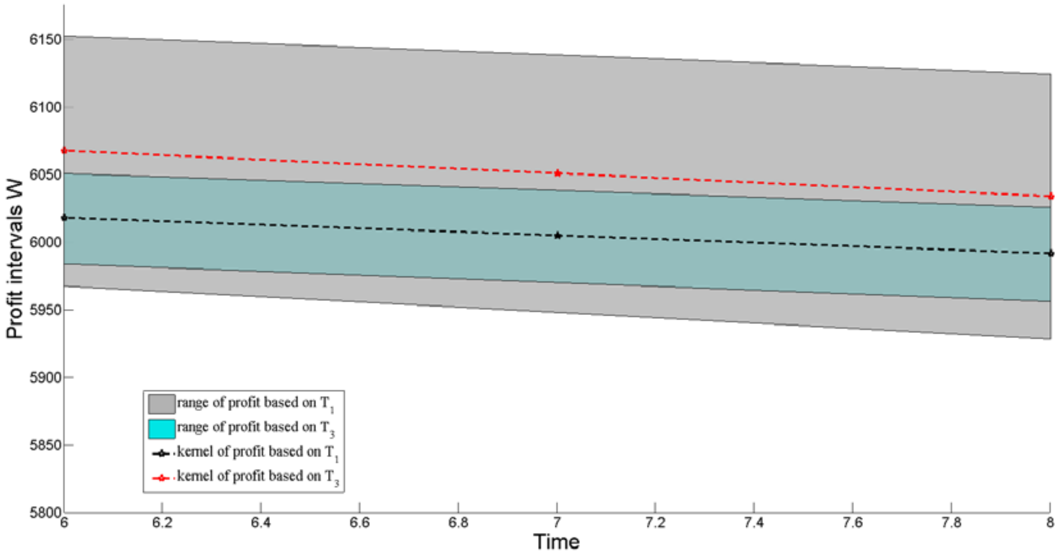


Fig.6. The expected ranges of pure profit in Sub-case 1.

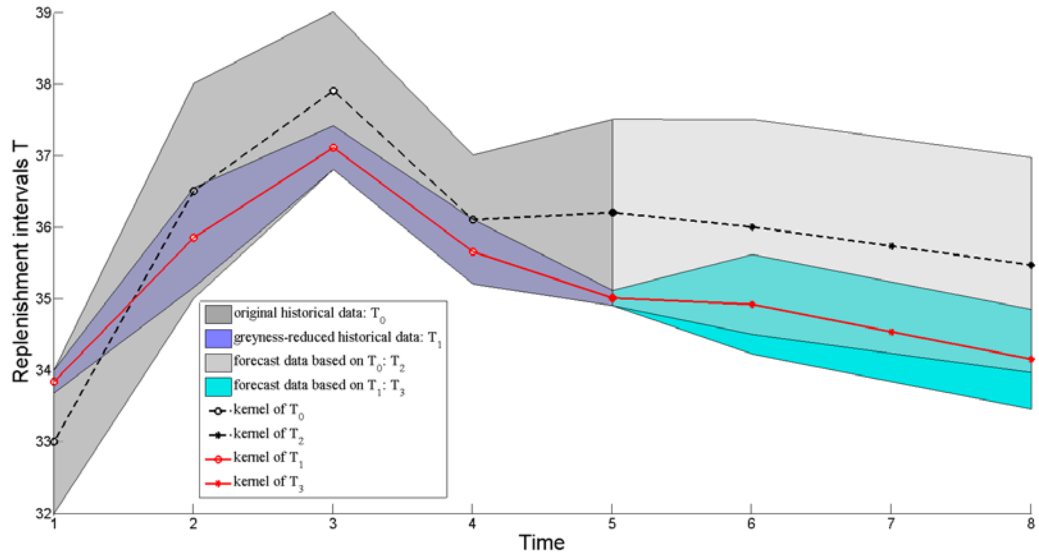


Fig.7. Replenishment intervals in Sub-case 2.

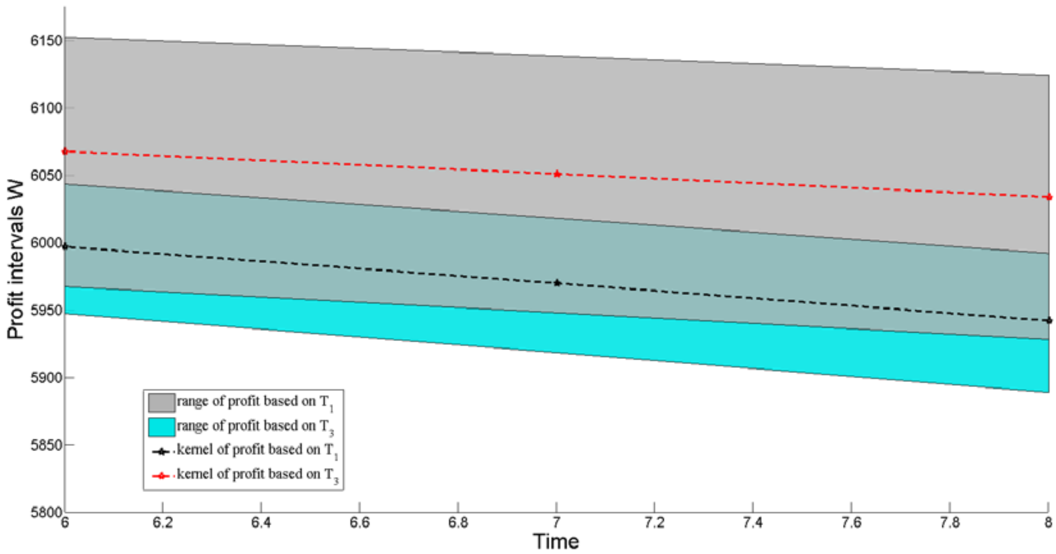


Fig.8. The expected ranges of pure profit in Sub-case 2.

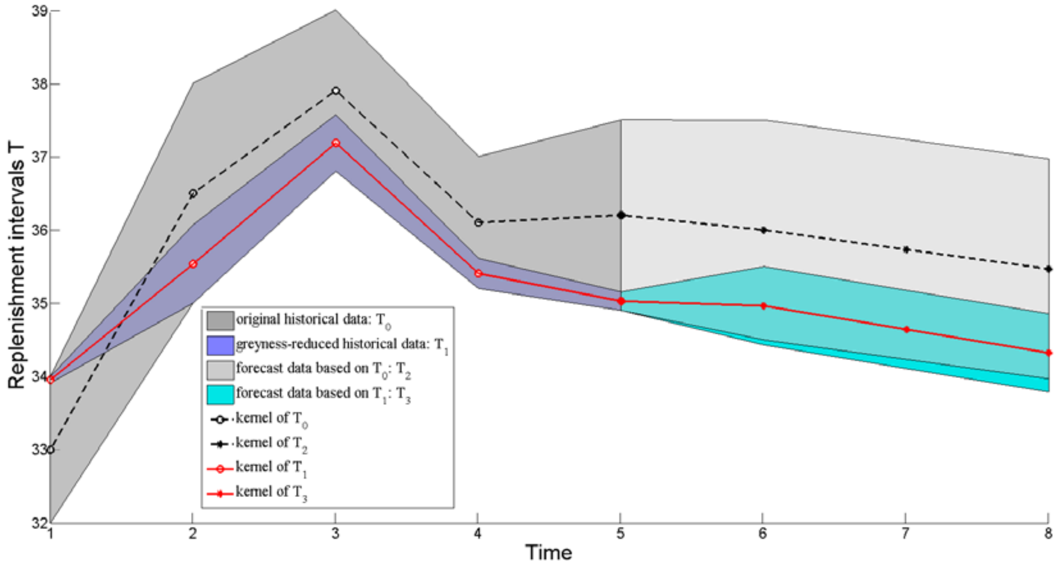


Fig.9. Replenishment intervals in Sub-case 3.

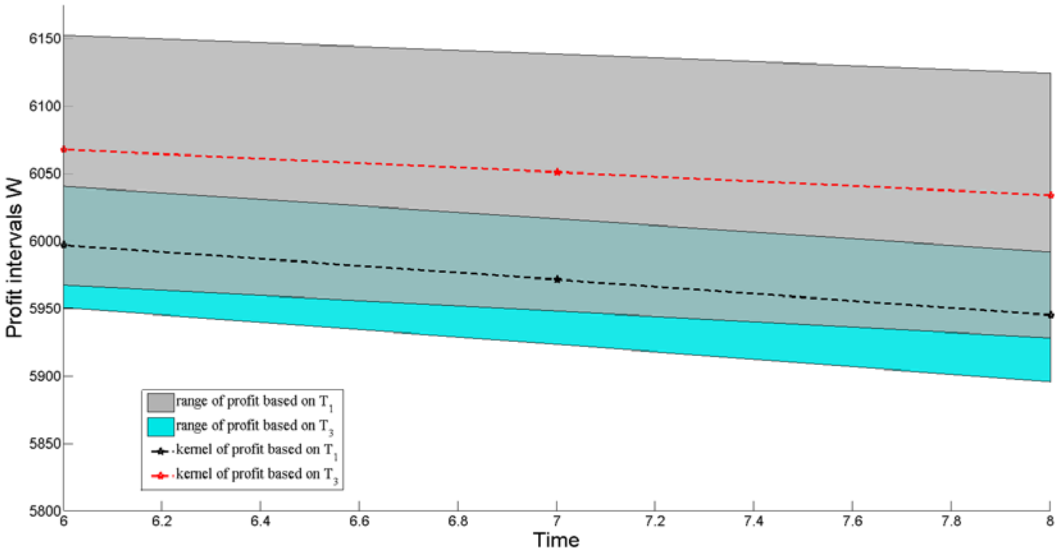


Fig.10. The expected ranges of pure profit in Sub-case 3.

First, comparing results of three sub-cases shows that the change of grey number parameters used in greyness reduction operation does influence the modeling samples, which will future affect the possible optimal replenishment quantity in the prediction period. Different from the other two sub-cases, the greyness reduction-based prediction intervals in sub-case 1 all fall into the prediction area of the original data. In addition, sub-case 1 has the best greyness reduction performance and the highest expected total pure profit range among three sub-cases. In sub-case 2 and sub-case 3 (Figs. 7 and 9), the new lower prediction bounds is slightly lower than the old one. Due to the deviation (uncertainty) of the replenishment interval data, the kernels of their expected pure profit are lower than that in sub-case 1.

Second, comparing sub-cases with the original case (Figs. 3 and 4), it is clear that all cases share similar prediction results. This means the proposed method is not sensitive to parameter values and its prediction result is stable. During the modeling period, when the original irregular value deviations of the production capability of upstream suppliers parameter h or the traffic condition parameter θ are replaced with some regular fluctuations (smoother or monotonous decline) in sub-cases, the greyness reduction effect increases: the range of uncertainties becomes smaller in all sub-cases and the negative impact of the abnormal large fluctuation at time point 3 is better reduced. These increase the prediction performance in three sub-cases.

The above cases imply that the greyness-reduced replenishment control information is more accurate and reliable. Prediction results with greyness reduction preprocessing also show good model stability. In practice, the proposed interval grey number preprocessing method can improve results of grey number prediction models and reduce negative effects of uncertain samples on decision-makings.

6. Conclusion

This paper provides new operational rules for multiplication and division of grey interval numbers. Compared with classic operational rules, the proposed calculation rules can make better use of potential information provided by the mathematical relationship between grey numbers and reduce the greyness degrees of results. Then the greyness reduction rules is integrated as a part of data preprocessing in a new prediction framework of grey heterogeneous interval numbers. Application of the proposed prediction framework to inventory optimization problem in emergency rescue scenarios shows that, the greyness reduction preprocessing can not only reduce the variation of predicted values but also improve the comparability of strategy alternatives that are based on prediction.

The proposed greyness reduction framework is rather general and can be applied to solve other economic and social problems. In future research, it may be worthwhile to study the operation rules of grey numbers more systematically from a theoretical point of view, so that they are applicable to operations of grey numbers with different discourse field.

Acknowledgement

This work was supported by the Leverhulme Trust International Network project entitled “Grey Systems and Its Applications”(IN-2014-020), the Natural Science Foundation of China (No. 71401039), the China Scholarship Council (No. 2016 06655020), the Humanities and Social Sciences Youth Fund Project of Education Ministry of China (No. 14YJC630060), and the Fujian Natural Science Foundation (No. 2017J01517).

Compliance with Ethical Standards

* **Funding:** This study was funded by the Leverhulme Trust International Network project entitled “Grey Systems and Its Applications”(grant number IN-2014-020), the Natural Science Foundation of China (grant number 71401039), the China Scholarship Council grant number 2016 06655020), the Humanities and Social Sciences Youth Fund Project of Education Ministry of China (grant number 14YJC630060), and the Fujian Natural Science Foundation (grant number 2017J01517).

* **Conflict of Interest:** Author Chong Li declares that she has no conflict of interest. Author Yingjie Yang declares that he has no conflict of interest. Author Sifeng Liu declares that he has no conflict of interest.

* **Ethical approval:** This article does not contain any studies with human participants or animals performed by any of the authors.

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